Lecture 12 - Oscillations

A Puzzle...

Differential equations appear all the time in physics (in fact, every equation $F = m\, a$ we have written has been a differential equation in x, since $a = \frac{d^2 x}{dt^2}$). In this class, knowledge of differential equations is optional (although assumed); this means that for this course you can simply double check that each solution to a differential equation presented to you actually works, and you will not need to solve any new differential equations in an exam. However, differential equations show up time and again *in every field*, so learning some tricks will be highly beneficial.

Differential Equation Boot Camp

Find the most general solution to the follow differential equation. Assume a, b, and c are constants that do not depend on x.

1. $y'[x] = a$
2. $y'[x] = a\, y[x]$
3. $y'[x] = a\, y[x] + b$
4. $a\, y''[x] + b\, y'[x] + c\, y[x] = 0$

Solution

Our purpose here is to show you some basic tricks used to solve differential equations. We discuss a more rigorous framework in the Differential Equations section below.

1. Integrating both sides yields $y[x] = a\, x + c_0$ where c_0 is an arbitrary constant. Note that the differential equation has degree 1 (i.e. the highest derivative which appears in the equation is the first derivative) and that there is 1 arbitrary constant in our solution. When these two numbers match, you have found the most general solution for the differential equation.

2. Rewrite $y'[x] = \frac{dy}{dx}$ and multiply both sides by $\frac{dx}{y}$ to obtain

$$\frac{dy}{y} = a\, dx \tag{1}$$

Integrating yields

$$\mathrm{Log}[y] = a\, x + c_0 \tag{2}$$

or equivalently

$$y[x] = c_1\, e^{a\, x} \tag{3}$$

where $c_1 = e^{c_0}$ is an arbitrary constant. The differential equation has order 1, and our solution has 1 arbitrary constant c_1, so we have found the most general solution.

3. From Part 2, we know that if $b = 0$, we know the solution would be $y[x] = c_1\, e^{a\, x}$. What if we tried adding a constant C to this solution,

$$y_{\text{guess}}[x] = c_1\, e^{a\, x} + C \tag{4}$$

Substituting this back into the differential equation,

$$c_1\, a\, e^{a\, x} = \frac{d y_{\text{guess}}[x]}{dx} = a\, y_{\text{guess}}[x] + b = a\,(c_1\, e^{a\, x} + C) + b \tag{5}$$

This equation will be satisfied if $C = -\frac{b}{a}$, so that the full solution is given by

$$y[x] = c_1 e^{a x} - \frac{b}{a} \tag{6}$$

Note that differential equation has order 1, and our solution has 1 arbitrary constant c_1, so we have found the most general solution. Taking a known solution and adding a constant to it is often a useful trick.

4. Based on Part 2, we might guess that the solution will be in the form of an exponential function, $y_{\text{guess}}[x] = C e^{m x}$. Substituting into the differential equation,

$$a C m^2 e^{m x} + b C m e^{m x} + c C e^{m x} = 0 \tag{7}$$

Factoring out the $C e^{m x}$,

$$a m^2 + b m + c = 0 \tag{8}$$

which is a simple quadratic equation in terms of m which has the two solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a} \tag{9}$$

Note that the C canceled, which means that it is a free parameter in each of our two solutions. Therefore, the most general solution to the differential equation equals

$$y[x] = c_1 e^{\frac{-b+\sqrt{b^2-4 a c}}{2 a} x} + c_2 e^{\frac{-b-\sqrt{b^2-4 a c}}{2 a} x} \tag{10}$$

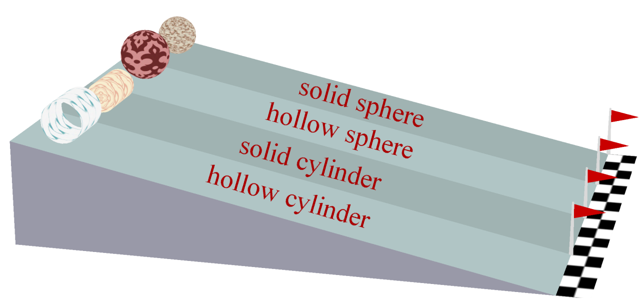
This time the differential equation has order 2 (since it features $\frac{d^2 y}{dx^2}$), and our solution has 2 arbitrary constants c_1 and c_2, so we have found the most general solution. □

Rotation Wrap Up

Race to the Finish!

Example

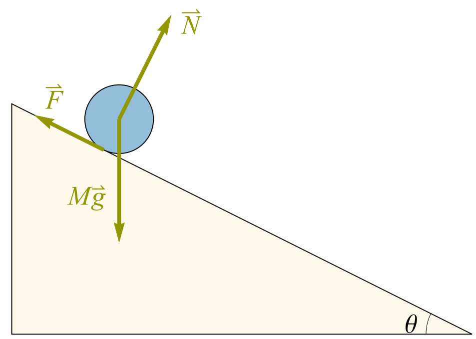
Four objects - a hollow cylinder, a solid cylinder, a hollow sphere, and a solid sphere - are placed on top of an incline and released from rest. In what order will the objects cross the finish line?



Solution

We seem to be missing a lot of information. For example, we do not know the masses or radii of any of the objects, nor do we know the angle θ of the incline. Therefore, these quantities must ultimately cancel out in the final answer.

One attribute that is the same about all four objects is that they all have a circular cross section when viewed from the side, so the analysis for all four objects will proceed identically. There are only three forces in the problem: gravity, friction, and a normal force.



Aligning a to be positive down along the incline and α to be positive clockwise, the sum of forces yields

$$M\,g\,\mathrm{Sin}[\theta] - F = M\,a \tag{11}$$

while the sum of torques about the center of mass becomes

$$R\,F = I\,\alpha = \eta\,M\,R^2\,\alpha \tag{12}$$

where we have written the moment of inertia as $I = \eta\,M\,R^2$. η is a constant that depends on the mass distribution or shape of an object. Recall from the last lecture that $\eta = 1, \frac{1}{2}, \frac{2}{3}, \frac{2}{5}$ for the hollow cylinder, solid cylinder, hollow sphere, and solid sphere, respectively.

Finally, the condition for rolling without slipping is

$$a = \alpha\,R \tag{13}$$

Substituting in this condition into Equation (12), we find

$$F = \eta\,M\,a \tag{14}$$

which we insert into Equation (11) to find

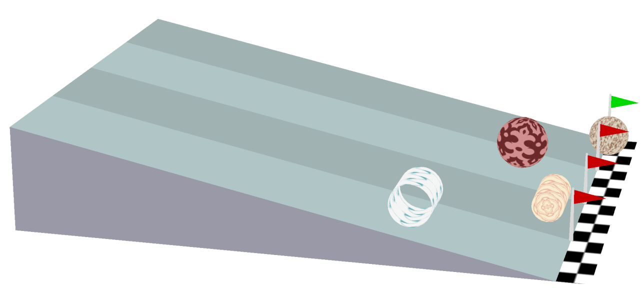
$$g\,\mathrm{Sin}[\theta] - \eta\,a = a \tag{15}$$

or equivalently

$$a = \frac{g\,\mathrm{Sin}[\theta]}{\eta+1} \tag{16}$$

This is an astonishing result. It suggest that regardless of the masses or radii of the four objects, their order at the finish line will be dictated solely by their moment of inertia, where the object with the smallest moment of inertia finishes first.

Therefore, we expect that the solid sphere will take 1st place, followed by the solid cylinder in 2nd place, the hollow sphere in 3rd place, and the hollow cylinder in 4th place. Indeed, this ordering can be seen in the snapshot below which shows the situation right as the solid sphere crosses the finish line.



Wikipedia has a beautiful animation of this same phenomenon together with other interesting examples of how the moment of inertia has been used in constructing various objects.

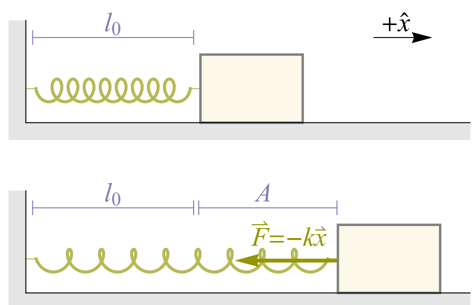
Oscillations

Supplementary Section: The Spring

The canonical example of simple harmonic motion is the horizontal oscillating spring. It is important to spend some time and truly understand this system.

Example

A mass is connected to a wall by a spring with spring constant k and natural spring length l_0. The mass rests on a frictionless horizontal table and is offset to a position $l_0 + x$ from the wall, where it is released from rest. What is the motion of the mass?



Solution

The spring is uncompressed and exerts zero force when $x = 0$. When x is non-zero, the force exerted by the spring equals $F = -k\,x$, where the force points in the direction that restores the spring to its unstretched length. Using $F = m\,a = m\,\ddot{x}$,

$$m\,\ddot{x} = -k\,x \tag{17}$$

Defining the *natural frequency* $\omega^2 = \frac{k}{m}$, we can rewrite this equation as

$$\ddot{x} = -\omega^2\,x \tag{18}$$

The problem is now reduced to solving this differential equation with the initial conditions $x[0] = A$ and $\dot{x}[0] = 0$. We mention two ways to solve this differential equation.

Solution 1

We can guess a solution of the form $x[t] = c_0\,e^{\alpha\,t}$ and see if it will solve our differential equation. Since $\ddot{x} = c_0\,\alpha^2\,e^{\alpha\,t}$, substituting in yields

$$c_0\,\alpha^2\,e^{\alpha\,t} = -\omega^2\,c_0\,e^{\alpha\,t} \tag{19}$$

which implies that $\alpha^2 = -\omega^2$ or equivalently $\alpha = \pm\,i\,\omega$. Note that the c_0 factor dropped out, indicating that it truly could have been any arbitrary constant. Thus we find the two solutions

$$x[t] = c_1\,e^{i\,\omega\,t} = c_1\,\mathrm{Cos}[\omega\,t] + i\,c_1\,\mathrm{Sin}[\omega\,t] \tag{20}$$
$$x[t] = c_2\,e^{-i\,\omega\,t} = c_2\,\mathrm{Cos}[\omega\,t] - i\,c_2\,\mathrm{Sin}[\omega\,t] \tag{21}$$

As discussed in the Advanced Section above, these two solutions can be added and comprise the most general solution to this system of equations.

$$x[t] = (c_1 + c_2)\,\mathrm{Cos}[\omega\,t] + i\,(c_1 - c_2)\,\mathrm{Sin}[\omega\,t] \tag{22}$$

Don't be alarmed by the fact that there is the imaginary i in this problem. It will go away when we solve for the initial conditions ($c_1 = \frac{A}{2}$ and $c_2 = \frac{A}{2}$) and we will obtain

$$x[t] = A\,\mathrm{Cos}[\omega\, t] \tag{23}$$

Technical Note

Another way you can think about this solution is to define the new constants $c_3 = c_1 + c_2$ and $c_4 = i\,(c_1 - c_2)$; just be aware when making such transformations that you do not accidentally take away a degree of freedom from this system. In this case, this transformation is fine because $c_1 + c_2$ and $i\,(c_1 - c_2)$ are not linearly dependent, so we are exchanging the two independent parameters c_1 and c_2 for the independent parameters c_3 and c_4.

Solution 2

We can solve this system directly using $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{d\dot{x}}{dx}\dot{x}$. Thus

$$\ddot{x} = \frac{d\dot{x}}{dx}\dot{x} = -\omega^2\, x \tag{24}$$

Separating out the variables and integrating

$$\dot{x}\, d\dot{x} = -\omega^2\, x\, dx \tag{25}$$

$$\frac{1}{2}\dot{x}^2 = -\frac{1}{2}\omega^2\, x^2 + \frac{c_1}{2} \tag{26}$$

where we have chosen the constant $\frac{c_1}{2}$ to get all of the $\frac{1}{2}$ factors to simplify. Rearranging terms and integrating,

$$\dot{x} = \frac{dx}{dt} = (c_1 - \omega^2\, x^2)^{1/2} \tag{27}$$

$$\frac{dx}{(c_1 - \omega^2\, x^2)^{1/2}} = dt \tag{28}$$

$$\frac{1}{\omega}\,\mathrm{ArcTan}\left[\frac{\omega\, x}{(c_1 - \omega^2\, x^2)^{1/2}}\right] = t + \frac{c_2}{\omega} \tag{29}$$

where we have again chosen the clever constant $\frac{c_2}{\omega}$. Therefore, we find the solution

$$\frac{\omega\, x}{(c_1 - \omega^2\, x^2)^{1/2}} = \mathrm{Tan}[\omega\, t + c_2] \tag{30}$$

which can be solved for x to obtain

$$x[t] = \frac{\sqrt{c_1}}{\omega}\,\mathrm{Sin}[\omega\, t + c_2] \tag{31}$$

As discussed in the following section, this form of the solution is exactly the same as the form $x[t] = c_3\,\mathrm{Cos}[\omega\, t] + c_4\,\mathrm{Sin}[\omega\, t]$ obtained above in Solution 1. As expected, once we substitute the initial conditions we find $c_1 = A^2\,\omega^2$ and $c_2 = \frac{\pi}{2}$ and obtain

$$x[t] = A\,\mathrm{Cos}[\omega\, t] \tag{32}$$

just like above. □

In case you did not approve of using the word "guessing" in Solution 1, there are various general methods that you can use to find solutions to differential equations. One such method is to insert a generic power series $\sum_{j=-\infty}^{\infty} c_j\, x^j$, obtain a relation between the c_j's, and hope that the ultimately solution will simply from an infinite series into a nice trig function such as Sin, Cos, e...

Equivalence of Trig Functions

As found in the last section, when solving differential equations in multiple ways, one may often encounter multiple solutions that look seemingly different but are actually the same. In particular, the following forms are all equivalent.

$$x[t] = A\, e^{i\,\omega\, t} + B\, e^{-i\,\omega\, t} \tag{33}$$

$$x[t] = C\,\mathrm{Cos}[\omega\, t] + D\,\mathrm{Sin}[\omega\, t] \tag{34}$$

$$x[t] = E\,\mathrm{Cos}[\omega\, t + \phi_1] \tag{35}$$

$$x[t] = F\,\mathrm{Sin}[\omega\, t + \phi_2] \tag{36}$$

The two constants in any of these equations are related to the two constants in all other equations (recall that

$e^{iz} = \text{Cos}[z] + i\,\text{Sin}[z]$ for any z). For example,

$$A = \tfrac{1}{2}(C - i\,D),\ B = \tfrac{1}{2}(C + i\,D) \tag{37}$$
$$C = E\,\text{Cos}[\phi_1],\ D = -E\,\text{Sin}[\phi_1] \tag{38}$$
$$E = F,\ \phi_1 = \phi_2 - \tfrac{\pi}{2} \tag{39}$$
$$F = 2\sqrt{A\,B},\ \phi_2 = \pi + \tfrac{i}{2}\text{Log}\left[\tfrac{A}{B}\right] \tag{40}$$

You do not need to memorize any of these relations; just understand that there are multiple forms to the most general solution for a problem.

Technical Note

As long as two solutions to an n^{th} order differential equation both have n independent parameters, they will be equivalent solutions.

Limits

A differential equation such as

$$\ddot{x} = a\,x \tag{41}$$

has "three different solutions" depending on whether $a > 0$, $a = 0$, or $a < 0$. These solutions are

$$\begin{aligned} &x = c_1\,\text{Cos}\left[\sqrt{-a}\ t\right] + c_2\,\text{Sin}\left[\sqrt{-a}\ t\right] && (a < 0) \\ &x = c_3 + c_4\,t && (a = 0) \\ &x = c_5\,e^{-\sqrt{a}\ t} + c_6\,e^{\sqrt{a}\ t} && (a > 0) \end{aligned} \tag{42}$$

A natural question to ask is: When a negative a approaches 0, does the solution $c_1\,\text{Cos}\left[\sqrt{a}\ t\right] + c_2\,\text{Sin}\left[\sqrt{a}\ t\right]$ approach the form $c_3 + c_4\,t$? The answer is yes, and in fact, it holds for all sufficiently nice differential equations (as discussed below in Advanced Section: Linear Differential Equations).

Let's consider a negative a and ask what happens as it approaches 0. The first thing to note is that the limit $a \to 0$ is actually very poor notation, because a has units of $[\text{time}]^{-2}$ while 0 has no units at all. It is much better to take the limit as $\sqrt{-a}\ t \to 0$ or $-a\,t^2 \ll 1$. In this limit, we can use the Taylor series of $\text{Cos}[z] \approx 1 + O[z^2]$ and $\text{Sin}[z] \approx z + O[z^3]$ to obtain

$$x \approx c_1 + c_2\sqrt{a}\ t \tag{43}$$

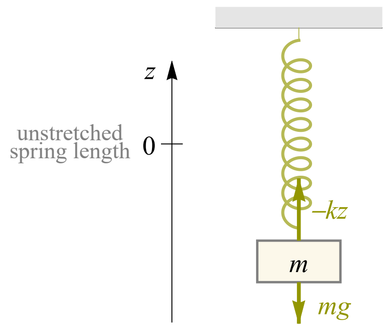
which we see does resemble the expected form $x = c_3 + c_4\,t$ at $a = 0$ if we define $c_3 = c_1$ and $c_4 = c_2\sqrt{a}$. Note that this relation only holds while $\sqrt{-a}\ t^2 \ll 1$, and for large enough times $1 \ll \sqrt{-a}\ t^2$ we can no longer use the Taylor series approximation and the two forms will diverge. However, as a gets closer to 0, the range of time for which $\sqrt{-a}\ t^2 \ll 1$ is valid increases, and when $a = 0$ this range becomes infinitely long.

Of course, the same case holds in the limit of a positive a approaching zero, analogously to the case shown above!

Supplementary Section: Hanging from the Ceiling

Example

A spring with spring constant k hangs unstretched. A mass m is attached to the free end of the string and then released from rest. What is the resulting motion of the mass?



<u>Solution</u>

Denote the position of the unstretched spring as $z = 0$. The equilibrium position (where the gravitational force $m\,g$ and the spring force $-k\,z$ sum to zero) occurs at $z_0 = -\frac{m\,g}{k}$. The force at any height is given by

$$m\,\ddot{z} = F = -k\,z - m\,g = -k\,(z - z_0) \tag{44}$$

Since z_0 is a constant, $\dot{z}_0 = \ddot{z}_0 = 0$ so we can write the above equation

$$m\,\ddot{(z - z_0)} = -k\,(z - z_0) \tag{45}$$

which is harmonic motion in the variable $z - z_0$. Thus, the mass will oscillate about the equilibrium position $z = z_0$ (you can prove this formally by changing variables to $\tilde{z} = z - z_0$). Setting $\omega_0 = \left(\frac{k}{m}\right)^{1/2}$, the solution is given by

$$z[t] - z_0 = c_1\,\mathrm{Sin}[\omega_0\,t] + c_2\,\mathrm{Cos}[\omega_0\,t] \tag{46}$$

Since the mass was released from rest at $t = 0$, the initial conditions are $z[0] = 0$ and $\dot{z}[0] = 0$ so that $c_1 = 0$ and $c_2 = -z_0$,

$$z[t] = z_0(1 - \mathrm{Cos}[\omega_0\,t]) \tag{47}$$

so the mass will oscillate between $z = 0$ and $z = 2\,z_0 = -\frac{2\,m\,g}{k}$.

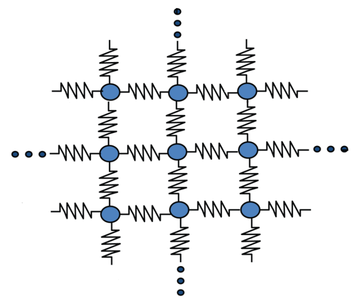
Note that this frequency of oscillation ω_0 is identical to that for a horizontal spring. This might seem surprising, but gravity exerts a constant force $m\,g$, which serves to simply shift the oscillations of the spring from being centered on $z = 0$ to $z = z_0$. Because the force exerted by the spring is linear, the resulting oscillations about $z = z_0$ are identical to the oscillations for the horizontal spring about $z = 0$. □

Advanced Section: Time Reversal

Angled Rails

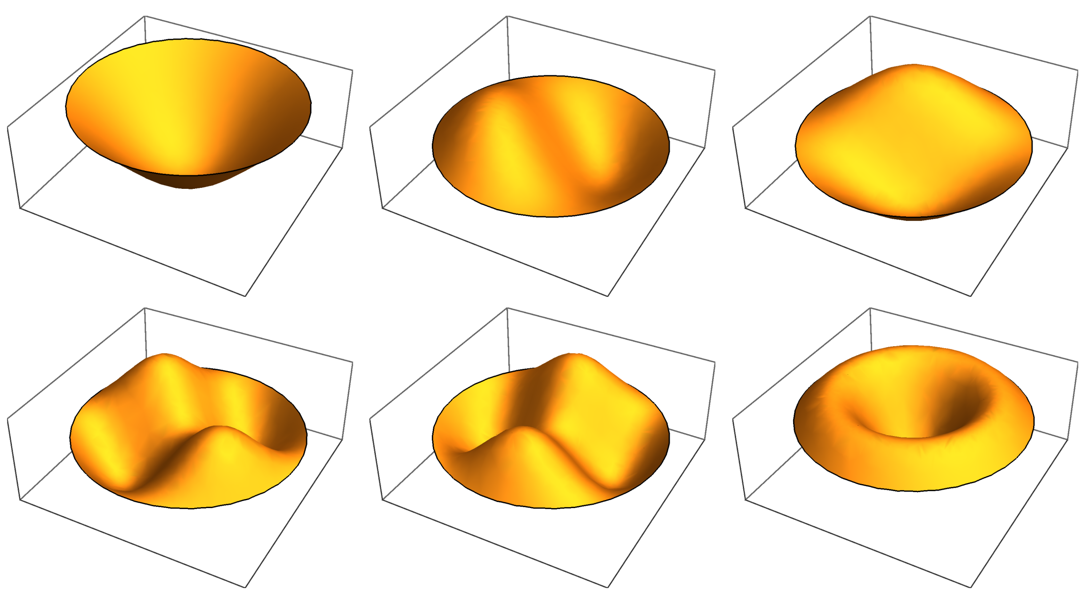
Real World Applications

So what have we learned so far? At first blush, it might not seem like much. We covered the motion of a mass on a spring and found out that this mass oscillates periodically. But it is easy to extend this simple analysis to many objects that we deal with in our daily lives. For example, a drum can be modeled as an array of masses connected by springs, where each mass oscillates into and out of the board.



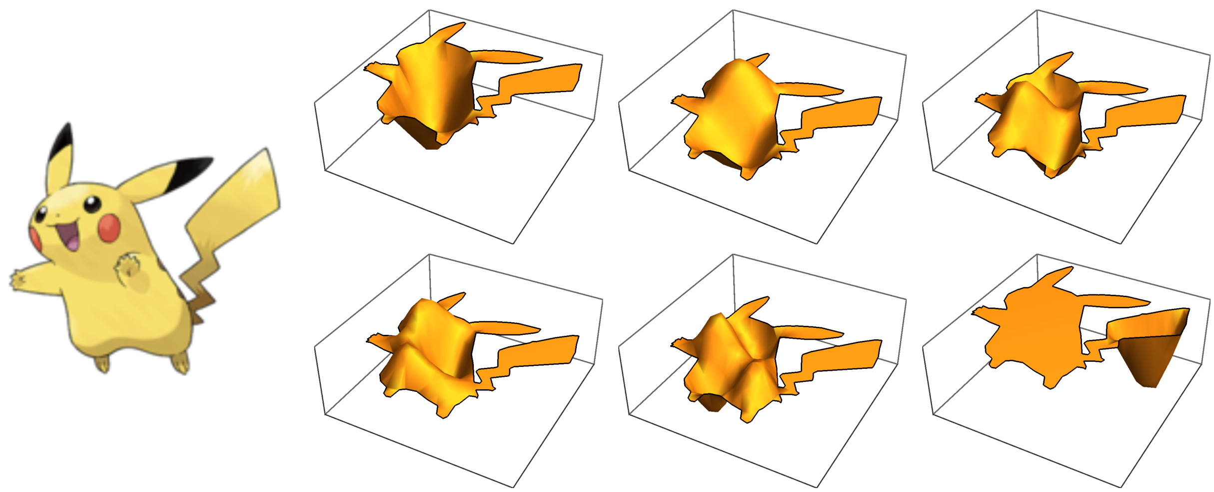
When you analyze such systems, you find that the particles collectively undergo what are called normal modes - it is essentially doing the wave on a large scale across the entire object. But there are many forms of the wave, and there are many normal modes. For a circular drum, the most prevalent normal modes are shown below. The fundamental mode (top-left) is exactly what you expect - the center of the drum goes up and down. But the other modes are weird, cool, and bizarre, with parts of the drum moving up while other parts move down (and then the two regions switch).

Normal Modes of a Drum



The drum is a great system, because its circular geometry is particularly simple, and hence the normal modes can be solved analytically, but if you are willing to go to numerics, you can solve the normal modes of any shape you want. For example, suppose you order a drum shaped like...a Pikachu! No problem!!! It takes 2 lines of Mathematica code to find the normal modes of this drum. From these normal modes, you can determine what the Pikachu drum will sound like (surprisingly, it does not emit a “pi-ka-chu” sound when struck).

Normal Modes of Pikachu



Next, let's talk about the normal modes of a rectangular plate. Just like in the drum example above, for each normal mode (except the fundamental node), some part of the rectangle will oscillate up while another part goes down, which necessarily implies that there is always some part of the rectangle that is not moving at all throughout the entire normal mode motion. This beautiful animation lets you visualize what these stationary contours look like.

Here is another extremely bizarre example. Five metronomes are ticking with the same frequency but at different phases. The metronomes are coupled together by putting all five of them on top of a wooden plank laid on two soda cans. Every time a metronome oscillates from side to side, it will make the wooden plank roll in the opposite direction (to keep the center of mass of the entire system stationary), and the other metronomes will feel that motion through the plank. Quite surprisingly, after a few seconds have elapsed, the metronomes are in sync!

There are many phenomenon that we encounter every day that are all based on oscillations, or can be modeled as oscillators. But let us take a step back and talk about physics. The mass on a spring - also known as the simple harmonic oscillator - is *the single most important model in all of physics!* I am not kidding. This system shows up repeatedly in every subject from quantum mechanics to condensed matter to quantum field theory. So believe it or not, in solving these simple problems today, we took a rather large step to understand how much of the universe works.

Advanced Section: Linear Differential Equations

Mathematica Initialization